

ON THE EQUALITY CASE IN EHRHART'S VOLUME CONJECTURE

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ABSTRACT. Ehrhart's conjecture proposes a sharp upper bound on the volume of a convex body whose barycenter is its only interior lattice point. Recently, Berman and Berndtsson proved this conjecture for a class of rational polytopes including reflexive polytopes. In particular, they showed that the complex projective space has the maximal anticanonical degree among all toric Kähler-Einstein Fano manifolds.

In this note, we prove that projective space is the only such toric manifold with maximal degree by proving its corresponding convex-geometric statement. We also discuss a generalized version of Ehrhart's conjecture involving an invariant corresponding to the so-called greatest lower bound on the Ricci curvature.

1. INTRODUCTION

Minkowski's famous lattice point theorem gives a sharp upper bound on the volume of a centrally-symmetric convex body containing only the origin as a strictly interior lattice point [11]. A *convex body* $K \subseteq \mathbb{R}^n$ is a compact convex set. K is *centrally symmetric* if $-x \in K$ for all $x \in K$. Two convex bodies K, K' in \mathbb{R}^n are *unimodularly equivalent*, if there is an affine lattice automorphism of \mathbb{Z}^n mapping K onto K' . Note that this transformation does not change the volume. Let Δ_n be the *n-dimensional unimodular simplex* with vertices $\mathbf{0}, e_1, \dots, e_n$, where $\mathbf{0} = (0, \dots, 0)$, and e_1, \dots, e_n is the standard lattice basis of \mathbb{Z}^n .

In 1964 Ehrhart [8] posed the following generalization of Minkowski's theorem as a conjecture (see also [3, E13] and [11]).

Conjecture 1.1 (Ehrhart '64). *Let $K \subseteq \mathbb{R}^n$ be an n-dimensional convex body with barycenter $\mathbf{0}$. If K contains only the origin as an interior lattice point, then*

$$\text{vol}(K) \leq (n+1)^n/n!,$$

where equality holds if and only if K is unimodularly equivalent to $(n+1)\Delta_n$.

Ehrhart proved the upper bound in the conjecture in several special cases, e.g., in dimension 2 [6] and for simplices [9]. The uniqueness of the equality case is suggested in [8] and proved in dimension 2 [7].

A very good upper bound which is only slightly weaker than the conjectured one can be deduced from a paper of Milman and Pajor. In Corollary 3(2) of [17] they show that $\text{vol}(K) \leq 2^n \text{vol}(K \cap (-K))$ for any compact convex body with barycenter in the origin. If additionally the origin is the only lattice point in the interior of K , then we can combine this result with Minkowski's first theorem to obtain $\text{vol}(K) \leq 4^n$. Let us note that by Stirling's approximation the conjectural upper bound is asymptotically equal to $e^{n+1}/\sqrt{2\pi n}$.

BN is supported by the US National Science Foundation (DMS 1203162).

AP is supported by the German Research Foundation (DFG SPP 1489).

For rational polytopes (i.e., polytopes with vertices in \mathbb{Q}^n), there is a natural relation of this conjecture to differential geometry via toric varieties, see e.g., [16, 20]. A polytope Q is a *lattice polytope* if its vertices are in \mathbb{Z}^n . Let the origin $\mathbf{0}$ be an interior point of Q . In this case, the *dual polytope* Q^* is defined as

$$Q^* := \{x \in \mathbb{R}^n : \langle v, x \rangle \geq -1 \text{ for all vertices } v \text{ of } Q\},$$

where we chose an inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^n (We remark that the usual convention is to consider dual vector spaces. We use the above notation for the sake of simplicity of this note.) It follows from Q being a lattice polytope that $\mathbf{0}$ is the only interior lattice point of the rational polytope Q^* . In general, Q^* is not a lattice polytope. If it is, then we say that Q and Q^* are *reflexive polytopes*.

For the following, let $Q \subseteq \mathbb{R}^n$ be a *Fano polytope*, i.e., an n -dimensional lattice polytope that contains the origin in its interior and whose vertices are primitive (i.e., not multiples of a non-zero lattice point). Then the associated fan over the faces of Q gives rise to a *toric Fano variety* X (see, e.g., [10] for details of this correspondence). In particular, $X \cong \mathbb{P}^n$ if and only if the dual of the associated polytope Q is unimodularly equivalent to $(n+1)\Delta_n$. Wang and Zhu [23] (in the nonsingular case) and Berman and Berndtsson [2] (in the singular case) showed that X admits a Kähler–Einstein metric if and only if the barycenter of Q^* is $\mathbf{0}$. Moreover, it is known that the anticanonical degree $c_1(X)^n$ equals $n! \operatorname{vol}(Q^*)$. Hence, Ehrhart’s conjecture can be phrased in this setting as the problem of finding the maximal degree of a Kähler–Einstein toric Fano variety [20]. Recently, this was solved by Berman and Berndtsson [2, Thm.1.2]:

Theorem 1.2 (Berman, Berndtsson ’12). *Let X be an n -dimensional toric Fano variety which admits a (singular) Kähler–Einstein metric. Then its first Chern class $c_1(X)$ satisfies the following upper bound:*

$$c_1(X)^n \leq (n+1)^n$$

The motivation of this note is to clarify the equality case:

Proposition 1.3. *In the situation of Theorem 1.2, $c_1(X)^n = (n+1)^n$ if and only if $X \cong \mathbb{P}^n$.*

This is a special case of the following main result.

Theorem 1.4. *The conclusion of Conjecture 1.1 holds for any n -dimensional convex body $K \subseteq \mathbb{R}^n$ with barycenter $\mathbf{0}$ that is contained in the dual polytope of a lattice polytope.*

The proof relies on the convex-geometric part of the argument in [2] which Berman and Berndtsson attribute to Bo’az Klartag. The only novel part is the treatment of the equality case. Note that the condition in the theorem is equivalent to the condition that the dual of K contains a full-dimensional lattice polytope having the origin in its interior. Here is an example of a convex body that does not satisfy this condition. Let $K := \operatorname{conv}(\pm(3/2, 1/4), \pm(3/2, 5/4))$. Then K is a convex body whose only interior lattice point is $\mathbf{0}$. Its dual is $K^* = \operatorname{conv}(\pm(1, -2), \pm(\frac{2}{3}, 0))$, and the only lattice points in K^* are $\mathbf{0}$ and $\pm(1, -2)$.

Theorem 1.4 generalizes Corollary 1.4 in [2]. In particular, Ehrhart’s conjecture holds for all reflexive polytopes with barycenter $\mathbf{0}$. This huge class of lattice polytopes is of importance in string theory since they give rise to many examples of mirror-symmetric Calabi–Yau manifolds [1, 18, 13, 14].

Organization of the paper: Section 2 contains the proof of Theorem 1.4. In Section 3 we discuss a variant of Conjecture 1.1 regarding the so-called greatest lower bound on the Ricci curvature.

Acknowledgments: We would like to thank Robert Berman and Xiaodong Wang for their interest and for telling us about the generalized version of Ehrhart's conjecture. We thank the referee for several suggestions to improve the paper. In particular, he pointed us to the result of Milman and Pajor mentioned above and its relation to Ehrhart's conjecture. We are especially grateful to Stanislaw Szarek for providing the alternative proof in Remark 2.3 and publishing his proof of Grünbaum's inequality as a courtesy to the readers.

2. PROOF OF THEOREM 1.4

We may assume $n \geq 2$. Let $\mathbf{0}$ be the origin in \mathbb{R}^n and $\mathbf{1} := (1, \dots, 1)$. We follow the approach in the proof of Corollary 1.4 and Remark 3.2 in [2]. A key ingredient of the proof is Grünbaum's inequality [12]. For any convex body $K \subseteq \mathbb{R}^n$ and any closed half-space H that contains the barycenter of K it states that

$$(1) \quad \text{vol}(K \cap H) \geq \left(\frac{n}{n+1} \right)^n \text{vol}(K).$$

Now let $K \subseteq \mathbb{R}^n$ be an n -dimensional convex body with barycenter $\mathbf{0}$ that is contained in $P := Q^*$ for a lattice polytope Q containing $\mathbf{0}$ in its interior. Fix a vertex v of P . The facets F of P are in one-to-one correspondence with the vertices l_F of Q . Choose facets F_1, \dots, F_n containing v such that l_{F_1}, \dots, l_{F_n} span \mathbb{R}^n . We consider the invertible affine-linear map

$$\begin{aligned} \phi : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto (\langle l_{F_1}, x \rangle + 1, \dots, \langle l_{F_n}, x \rangle + 1). \end{aligned}$$

By construction, ϕ maps the polytope P into the non-negative orthant $\mathbb{R}_{\geq 0}^n$. Since l_{F_1}, \dots, l_{F_n} are lattice points we have $|\det(\phi)| \geq 1$. In particular,

$$(2) \quad \text{vol}(K) \leq |\det(\phi)| \text{vol}(\phi(K)) = \text{vol}(\phi(K)).$$

The barycenter of $\phi(K)$ is the image of the barycenter of K , so $b := \phi(\mathbf{0}) = \mathbf{1}$. Consider $\eta := \mathbf{1}$ as a linear functional on \mathbb{R}^n . Then $\eta^- := \{x \in \mathbb{R}^n : \langle \eta, x \rangle \leq n\}$ is an affine half-space containing the origin in its interior and b on the boundary. Using $\phi(K) \subseteq \phi(P) \subseteq \mathbb{R}_{\geq 0}^n$, we get

$$\begin{aligned} (3) \quad \left(\frac{n}{n+1} \right)^n \text{vol}(\phi(K)) &\leq \text{vol}(\phi(K) \cap \eta^-) \leq \text{vol}(\phi(P) \cap \eta^-) \\ &\leq \text{vol}(\mathbb{R}_{\geq 0}^n \cap \eta^-) = \text{vol}(n\Delta_n) = \frac{n^n}{n!}, \end{aligned}$$

where the first inequality follows from (1). Combining this with (2) implies the desired inequality $\text{vol}(K) \leq \text{vol}(\phi(K)) \leq (n+1)^n/n!$.

Now, let us assume that $\text{vol}(K) = (n+1)^n/n!$. This implies that all inequalities in (2) and (3) are in fact equalities. The equations in (3) imply $\mathbb{R}_{\geq 0}^n \cap \eta^- = \phi(K) \cap \eta^-$. Hence, $\phi(v) = \mathbf{0} \in \mathbb{R}_{\geq 0}^n \cap \eta^-$ yields $v \in K$. Since v was chosen arbitrarily we get $K = P$. Moreover, equality in (2) implies that $|\det(\phi)| = 1$, so ϕ is an affine lattice transformation of \mathbb{Z}^n . Thus, P and $\phi(P)$ are unimodularly equivalent. In particular, since $\mathbf{0}$ is a lattice point, also v is a lattice point. Again, since v was arbitrary, P is a lattice polytope, and therefore reflexive (recall that by assumption $P^* = Q$ is a lattice polytope).

We will use the following result on reflexive polytopes contained in the paper [19] of the first author:

Lemma 2.1. *Let S be a reflexive polytope with barycenter in the origin. If m is a relatively interior lattice point of a facet of S , then $-m$ is again in the relative interior of a facet of S .*

Proof. This is precisely the implication from i. to (a) in part (2) of Theorem 5.2 in [19]. In that paper P being semisimple means that the set of roots is centrally-symmetric; and the set of roots is the precisely the set of lattice points in the relative interior of facets of P , see Definition 4.1 in [19] and the paragraph before. \square

It remains to consider the following situation: We have an n -dimensional polytope R (which is the polytope $\phi(P)$ above) with $\text{vol}(R) = \text{vol}((n+1)\Delta_n)$ such that

- (1) the barycenter b_R equals $\mathbf{1}$.
- (2) $n\Delta_n \subseteq R$ and they share the vertex cone at $\mathbf{0}$. Thus e_1, \dots, e_n are primitive inner facet normals.
- (3) $R - \mathbf{1}$ is a reflexive polytope.

Lemma 2.2. *In this situation, we also have the following condition:*

- (4) for any $i = 1, \dots, n$, the point $\mathbf{1} + e_i$ is contained in the relative interior of a facet of R .

Proof. We may assume $i = 1$. Since $n\Delta_n \subseteq R$, we see that $\mathbf{1} - e_1$ is in the relative interior of the facet $F := \{x \in R : \langle e_1, x \rangle = 0\}$. Hence, we can apply Lemma 2.1 to $R - \mathbf{1}$ and $m = (\mathbf{1} - e_1) - \mathbf{1} = -e_1$ to see that e_1 is in the relative interior of a facet of $R - \mathbf{1}$. This proves the claim. \square

We will show that $R = (n+1)\Delta_n$. For $i = 1, \dots, n$, let us denote by G_i the unique facet of R containing $m_i := \mathbf{1} + e_i$. Furthermore, let u_i be the primitive outer normal of G_i . Translating a polytope does not change the normal fan, so, as $R - \mathbf{1}$ is reflexive, we obtain

$$(4) \quad \langle u_i, m_i - \mathbf{1} \rangle = \langle u_i, e_i \rangle = 1 \quad \text{and} \quad \langle u_i, m_j - \mathbf{1} \rangle = \langle u_i, e_j \rangle \leq 1 \quad \text{for } j \neq i$$

Let $u_1 := (c_1, \dots, c_n)$. Then (4) implies $c_1 = 1$ and $c_j \leq 1$ for $j = 2, \dots, n$. If $c_j = 1$ for some $j \geq 2$, then $m_j \in G_1$, and thus $G_j = G_1$.

By assumption we have $n\Delta_n \subseteq R$, so $\langle u_1, ne_1 - \mathbf{1} \rangle \leq 1$. Thus, for any $j \geq 2$

$$\begin{aligned} n &= \langle u_1, n \cdot e_1 \rangle \leq \langle u_1, \mathbf{1} \rangle + 1 = 2 + c_2 + c_3 + \dots + c_n \\ &\leq 1 + c_j + (n-1) = n + c_j. \end{aligned}$$

This implies that $c_j \geq 0$, so $c_j \in \{0, 1\}$ for $j = 2, \dots, n$. Furthermore, if $c_j = 0$ for some j , then $c_k = 1$ for all k except j .

Now assume that indeed there is some j with $c_j = 0$, say $j = 2$. Then $u_1 = \mathbf{1} - e_2$. We have observed above that $c_j = 1$ implies $G_j = G_1$, hence $G_1 = G_3 = \dots = G_n$. We consider $u_2 := (d_1, \dots, d_n)$. Similarly as for u_1 we obtain $d_2 = 1$ and $d_j \in \{0, 1\}$ for $j \neq 2$. Since $m_2 = \mathbf{1} + e_2 \notin G_1$ (because of $\langle u_1, m_2 - \mathbf{1} \rangle = 0 < 1$), we have $G_2 \neq G_j$ for $j \in \{1, 3, \dots, n\}$. Hence, again by the above remark, we must have $d_k \neq 1$ for any $k = 1, 3, \dots, n$. Therefore, $u_2 = e_2$, so $\langle u_1, m_1 \rangle = n$ and $\langle u_2, m_2 \rangle = 2$ yields

$$R \subseteq \{x \in \mathbb{R}_{\geq 0}^n : \langle \mathbf{1} - e_2, x \rangle \leq n, \langle e_2, x \rangle \leq 2\}.$$

Here, the right polytope is unimodularly equivalent to $(n\Delta_{n-1}) \times [0, 2]$. Therefore,

$$\frac{(n+1)^n}{n!} = \text{vol}(R) \leq \frac{2n^{n-1}}{(n-1)!},$$

which implies $(n+1)^n \leq 2n^n$, a contradiction for $n \geq 2$.

Therefore, $c_j = 1$ for $j = 2, \dots, n$. Hence, $u_1 = (1, \dots, 1)$. So,

$$R \subseteq \{x \in \mathbb{R}_{\geq 0}^n : \langle \mathbf{1}, x \rangle \leq n+1\} = (n+1)\Delta_n.$$

Since both polytopes have the same volume, we get equality. \square

Remark 2.3. As was pointed out to us by Szarek after this paper has appeared as a preprint, the proof can be substantially simplified by using a characterization of the equality case in Grünbaum's inequality. In the notation of the proof: *inequality (1) holds with equality if and only if K is a pyramid over a base spanning an affine hyperplane parallel to h , where h is the affine hyperplane bounding the half-space H .*

Here is how one applies this result in the equality case. Let $R = \phi(P)$ again with $\text{vol}(R) = \text{vol}((n+1)\Delta_n)$, barycenter $b_R = \mathbf{1}$, and $n\Delta_n \subseteq R$. From the previous result we deduce that R is a pyramid with apex $\mathbf{0}$ and whose base facet has outer facet normal $\mathbf{1}$. Since the n coordinate planes are supporting facet hyperplanes of R , this implies that R must be a simplex of the form $\lambda\Delta_n$ for some $\lambda \geq n$. From $\text{vol}(R) = \text{vol}((n+1)\Delta_n)$ we get $\lambda = n+1$, as desired.

At the time this paper was published as a preprint, the proof of the equality characterization of Grünbaum's inequality was not available in the literature. Moreover, the main result in Grünbaum's original paper [12] is stated slightly differently, and so a characterization of the equality case in that setting wouldn't be directly applicable to our situation. In 2003, Szarek found an alternative proof of Grünbaum's inequality that allows to deal more directly with the case of equality. He made his proof now publicly available [21].

3. GENERALIZED EHRHART'S CONJECTURE AND GREATEST LOWER BOUND ON THE RICCI CURVATURE

For a general Fano manifold X one can define

$$R(X) := \sup\{t \in [0, 1] : \text{Ric } \omega \geq t\omega \text{ for all Kähler metrics } \omega \in c_1(X)\},$$

giving a “greatest lower bound on the Ricci curvature”. See [15, 22] for background and motivation. In Theorem 2.9 of [2], Berman and Berndtsson gave (in more generality) the following generalization of Theorem 1.2 in the case that there is no Kähler-Einstein metric:

Theorem 3.1 (Berman, Berndtsson '12). *Let X be a toric Fano manifold. Then the first Chern class $c_1(X)$ satisfies the following upper bound*

$$c_1(X)^n \leq \left(\frac{n+1}{R(X)}\right)^n$$

As a convex-geometric analogue, Li [15] introduced the following invariant for a convex body K .

Definition 3.2. Let $K \subseteq \mathbb{R}^n$ be an n -dimensional convex body containing $\mathbf{0}$ in its interior. Let b_K be its barycenter. If $b_K = \mathbf{0}$, then we define $R(K) := 1$. Otherwise, let x_K be the intersection of $b_K - \mathbb{R}_{\geq 0}b_K$ with the boundary of K . Then $R(K)$ is defined as the distance of x_K from $\mathbf{0}$ divided by the distance of x_K from b_K . In particular, $R(K) < 1$ in this case.

Let Q be an n -dimensional Fano polytope with associated toric variety X_Q . In this case Li [15] showed that for smooth X_Q the two definitions of R coincide, i.e. $R(X) = R(Q^*)$.

Taking this invariant $R(K)$ into account, Xiaodong Wang suggested a natural generalization of Ehrhart's conjecture for convex bodies with arbitrary barycenter.

Conjecture 3.3. *Let $K \subseteq \mathbb{R}^n$ be an n -dimensional convex body. If K contains only the origin as an interior lattice point, then*

$$\text{vol}(K) \leq \frac{(n+1)^n}{n! R(K)^n},$$

where equality holds if and only if K is unimodularly equivalent to $(n+1)\Delta_n$.

We would like to remark that this apparent generalization is actually equivalent to Ehrhart's conjecture.

Proposition 3.4. *Conjecture 3.3 is equivalent to Conjecture 1.1.*

Proof. Let $K \subseteq \mathbb{R}^n$ be an n -dimensional convex body such that K contains only the origin as an interior lattice point. Let $r := R(K) < 1$, and b_K the barycenter of K .

We define $K' := r(K - b_K)$. We claim $K' \subseteq K$. For this, let $u \in \mathbb{R}^n$, $c \in \mathbb{R}$ such that $\langle u, x \rangle \leq c$ for any $x \in K$. Then $\langle u, r(x - b_K) \rangle \leq r(c - \langle u, b_K \rangle)$. It suffices to show that $r(c - \langle u, b_K \rangle) \leq c$. However, this is equivalent to $\langle u, (r/(r-1))b_K \rangle \leq c$ which holds by our assumption: $(r/(r-1))b_K = x_K \in K$.

Therefore, K' is an n -dimensional convex body with barycenter $\mathbf{0}$ which contains no other interior lattice points. Since $\text{vol}(K') = r^n \text{vol}(K)$, we have $\text{vol}(K') \leq (n+1)^n/n!$ if and only if $\text{vol}(K) \leq (n+1)^n/(n!r^n)$. It remains to consider the equality case. Assume $K' = (n+1)\Delta_n$. Since $K \supseteq K'$ and any facet of K' contains a lattice point in its relative interior, we have $K = K'$. \square

From the proof we get the following generalization of Theorem 1.4.

Corollary 3.5. *The conclusion of Conjecture 3.3 holds for any n -dimensional convex body $K \subseteq \mathbb{R}^n$ that is contained in the dual polytope of a lattice polytope.*

In particular, this implies a convex-geometric proof of Theorem 3.1. The following consequence seems to be new:

Corollary 3.6. *In the situation of Theorem 3.1, $c_1(X)^n = \left(\frac{n+1}{R(X)}\right)^n$ if and only if $X \cong \mathbb{P}^n$.*

Proof. Recall that $c_1(X)^n = n! \text{vol}(Q^*)$ and $R(X) = R(Q^*)$. Hence, if equality in Theorem 3.1 holds, then the previous corollary implies that Q^* is unimodularly equivalent to $(n+1)\Delta_n$, thus $X \cong \mathbb{P}^n$. On the other hand, if $X \cong \mathbb{P}^n$, then $c_1(X)^n = (n+1)^n$ and $R(X) = 1$. \square

Finally, let us remark that Debarre showed in [4] on p.139 that for a general toric Fano n -fold X there is no polynomial bound on $\sqrt[n]{c_1(X)^n}$. In particular, there are examples of toric Fano n -folds X_n such that $R(X_n) \rightarrow 0$ for $n \rightarrow \infty$.

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